

# Complex Networks: effect of subtle changes in nature of randomness

Sanchari Goswami,<sup>1</sup> Soham Biswas,<sup>1</sup> and Parongama Sen<sup>1</sup>

<sup>1</sup>*Department of Physics, University of Calcutta, 92 Acharya Prafulla Chandra Road, Kolkata 700009, India*

In two different classes of network models, namely, the Watts Strogatz type and the Euclidean type, subtle changes have been introduced in the randomness. In the Watts Strogatz type network, rewiring has been done in different ways and although the qualitative results remain same, finite differences in the exponents are observed. In the Euclidean type networks, where at least one finite phase transition occurs, two models differing in a similar way have been considered. The results show a possible shift in one of the phase transition points but no change in the values of the exponents. The WS and Euclidean type models are equivalent for extreme values of the parameters; we compare their behaviour for intermediate values.

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## I. INTRODUCTION

Ever since the discovery of small world effect in many natural and artificial networks, many models have been proposed to mimic their features [1]. Basic properties like the small world effect can be obtained from the Watts Strogatz (WS) model [2] while a prototype of the scale-free model is the Barabasi Albert model [3]. For networks in which links depend on the geographical distance or age separating the nodes, a few models have also been proposed [4, 5].

In all these models, an essential ingredient is randomness. Randomness is involved in the way the nodes are connected in the network. For example, in the original WS model, links between nearest neighbours were rewired with a certain probability to a randomly chosen node.

While randomness is the key feature, one may ask the question, to what extent does its nature affect the network? For example, one can have a variation of the WS model, in which links are added randomly keeping the original links intact [6]. Thus one can conceive of two kinds of WS models, the rewiring type and the addition type. The network properties are in general identical for the two types.

Another class of models which has been studied extensively includes the evolving networks. In these networks, a new node prefers to get attached to older nodes characterised by some particular feature like degree, age or geographical proximity etc. In an Euclidean network, links are established between nodes with a probability which depends on the Euclidean distance separating them. These probabilities can be tuned to change the nature of the randomness. Changing the degree of the preferential attachment in an evolving network has been shown to lead to drastic changes its behaviour [7]. However, this variation is much more severe compared to the example of addition and rewiring type WS models.

In this paper, we have considered two classes of models. One is the WS type with rewiring and the other class is Euclidean networks in which links are attached with the probability proportional to  $l^{-\alpha}$  where  $l$  is the Euclidean distance separating them. Within both classes, we now

introduce subtle changes in the randomness (to be elaborated in the appropriate sections) and see how it affects the (static) network properties. At one level, we discuss the effect induced by the change in randomness within each class and on a different level, we compare the WS and the Euclidean networks which are equivalent for extreme values of the parameters.

In section II, we briefly describe the models and the network properties which have been studied. The WS type networks have been discussed in section III and in section IV, the Euclidean network results have been presented. Comments and discussions are made in Section V.

## II. THE MODELS AND THE NETWORK PROPERTIES STUDIED

To construct the WS type of networks, we start with a regular network with  $2K = 4$  nearest neighbours. In the first type of WS network, henceforth referred to as WSA, we rewind both the first and second neighbours with probability  $p$ . In the other type, *only* the second neighbours are rewired and we call it the WSB.

WSB may be looked upon as a particular case of WS networks with  $2K$  original links for each node and where rewiring of the 2nd, 3rd... $K$ th links are done only, i.e., the first nearest neighbour links are kept intact. Such a case was recently considered with rather high values of  $K$  and the results were found to be identical with WS [8]. This result is not surprising, as the number of rewired links for large  $K$  is considerably larger than the number of links which are not rewired. In case of  $K = 2$ , i.e., the present case, however, the effect (if any) of this kind of partial rewiring on the network properties is expected to be much more prominent as the number of links which are being rewired and those which are not, are both equal to  $L$  (the number of nodes), in model WSB.

In the Euclidean network, nodes placed at a distance  $l$  along a one dimensional chain are linked with a probability  $P(l) \propto l^{-\alpha}$ . In the first type of model which we call Euclidean A type (EA), nodes at any  $l \geq 1$  are linked while in the second type, EB, the first nearest neighbours are always linked while other nodes are linked with the

probability  $P(l)$  with  $l \geq 2$ . A schematic picture of the networks is given in Fig. 1.

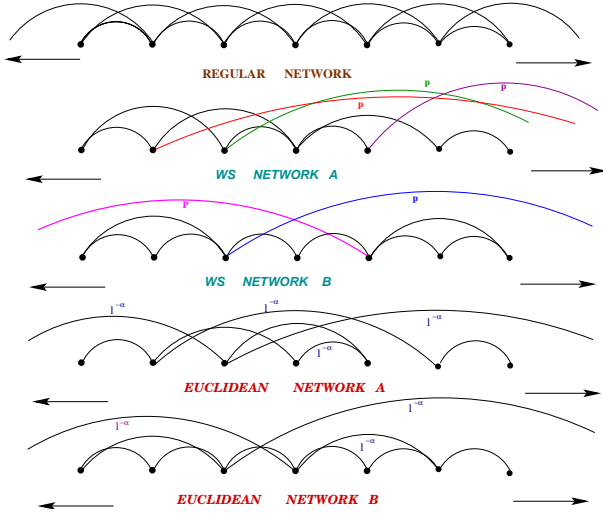


FIG. 1: (Color online) Schematic diagram for different network models. Average degree is  $2K = 4$  in each network. In the regular network both the first and second nearest neighbour are present. In WS network A both the neighbours are rewired while for WS network B only second neighbours are rewired with probability  $p$ . In Euclidean network A nodes at any  $l \geq 1$  are linked while in Euclidean network B, the first nearest neighbours are always linked while other nodes are linked with the probability  $l^{-\alpha}$  with  $l \geq 2$ .

In both WSB and EB, a broken network will never appear. The Euclidean network of type B was first studied in the context of navigation [4] and later in relation to polymers [9–11] where short cuts or bridges appear in a comparable manner. Other studies on the EB were later conducted where issues like clustering properties, navigation with local information and phase transition of the Ising model were addressed [12–16]. The results of the various earlier studies indicate that the EB model behaves as a regular network for  $\alpha > 2$  and like a small world for  $\alpha \leq 1$ . For  $1 < \alpha < 2$ , there is some controversy regarding the behaviour of the model; while some studies suggest that it is small world like, there are others which indicate a regular lattice (of dimension greater than one)-like behaviour [16]. In [12], it was claimed that there are two transition points and the region  $0 < \alpha < 1$  is a random graph with vanishing clustering tendency. In these studies, EB has been considered with two nearest neighbours plus one (on an average) long range link for each node. The main motivation for the present study comes from the fact that EA has never been studied earlier. Here we consider for EA, four random long range links for each node on an average and for EB, two nearest neighbour links plus two random long range links (on an average). This is done to make the Euclidean networks comparable with the Watts Strogatz like network for which the average degree is four.

The network properties we have studied are the basic ones like average shortest paths, clustering coefficient

and degree distribution. The average shortest paths have been calculated *exactly* using a burning algorithm.

The two types of models, WS and Euclidean, are equivalent and identical to a random network in the limit  $p = 1$  and  $\alpha = 0$ . They are also equivalent for  $p = 0$  and  $\alpha = \infty$ , corresponding to a regular chain of nodes with four nearest neighbour links for each node. For this regular network the average shortest path is

$$\langle s \rangle = \frac{(L + 3 - X)(L - 1 + X)}{8(L - 1)}, \quad (1)$$

where  $L$  is the number of nodes or system size and  $X$  is the remainder when  $(L - 1)$  is divided by 4.

Clustering coefficient of the  $i$ th node is defined by

$$C_i = \frac{T_i}{[k_i(k_i - 1)/2]},$$

where  $T_i$  is the number of closed triangles attached with the node  $i$  and  $k_i$  its degree. The average clustering coefficient of the regular network at  $p = 0$  or  $\alpha = \infty$  is 0.5.

Degree distribution is another important property of a network. Both the WS type and Euclidean networks are homogeneous, which means that the degree distribution will have a typical scale. For the regular network obtained in the the extreme limits  $\alpha \rightarrow \infty$  and  $p = 0$ , the degree distribution is simply a delta function at  $k = 4$ .

### III. WATTS STROGATZ TYPE NETWORKS

We have calculated the average shortest paths as a function of the number of nodes ( $L$ ) and rewiring probability ( $p$ ). For both WSA and WSB, the average shortest path varies as  $\ln(L)$  for  $p \neq 0$ . But when the variation against the rewiring probability  $p$  is studied, there appears to be some quantitative difference in the results between WSA and WSB.

It is now widely accepted that the average shortest path obeys the general scaling form [1]

$$\langle s \rangle \sim \frac{N^{1/d}}{K} f(KpN), \quad (2)$$

where  $N$  is the total number of nodes in a space of dimension  $d$  and

$$f(KpN) = \frac{\ln(KpN)}{KpN} \quad \text{for } KpN \gg 1.$$

In the present case, the dimension of the system  $d = 1$  and  $N = L$ , such that equation (2) becomes

$$\langle s \rangle \sim (1/K^2) \frac{\ln(p) + \ln(KL)}{p} \quad \text{for } KpL \gg 1. \quad (3)$$

However, from the numerical results for finite systems shown in Fig. 2, we find the following scaling form to be more appropriate

$$\langle s \rangle \sim a \frac{\ln(p) + c}{p^b}, \quad (4)$$

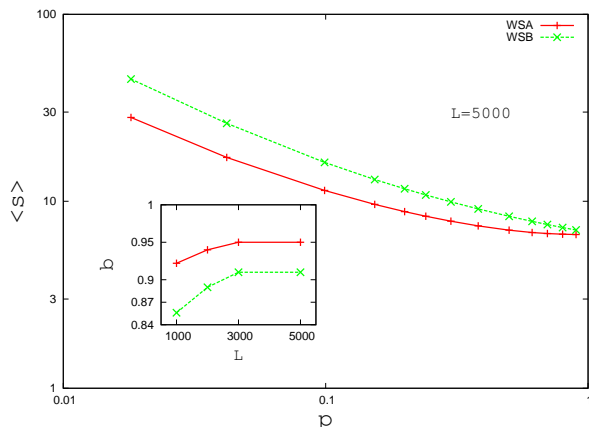


FIG. 2: (Color online) Variation of average shortest path as a function of rewiring probability  $p$  for the WSA and WSB models ( $L = 5000$ ). Inset shows the variation of  $b$  with system size.

where  $b$  is not exactly equal to unity and depends on the system size. The value of  $b$ , when plotted against  $L$ , indicates that it saturates at large values of  $L$ , with the saturation values close to but not exactly equal to unity (see inset of Fig. 2). Moreover, the saturation values of  $b$  ( $\sim 0.95$  for WSA and  $\sim 0.91$  for WSB) have a finite difference for the two models at least over the range of values of  $L$  considered here.

The average clustering coefficient for  $p = 0$  is 0.5 as mentioned before. For  $p > 0$ , two neighbors of a node  $i$  that were connected at  $p = 0$  are still neighbors of  $i$  and connected by an edge with probability  $(1 - p)$ . Since there are three edges that need to remain intact for a cluster,  $C(p) \simeq C(0)(1 - p)^3$  in WSA. For WSB  $C(p) \simeq C(0)(1 - p)$  as we do not rewire the first neighbours in this case. This is exactly what we observe in the numerical studies (Fig. 3). Hence WSA and WSB have clearly different exponents as  $p \rightarrow 1$ , the point at which the transition to random network is obtained.

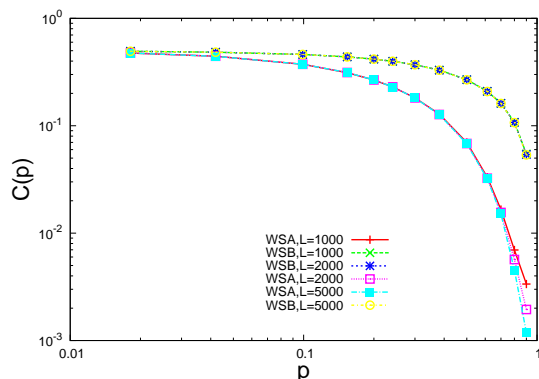


FIG. 3: (Color online) Average clustering coefficient  $C(p)$  as a function of rewiring probability  $p$  for the WSA and WSB model for different system sizes are shown.  $C(p)$  is fitted to the form  $C(p) \sim (1 - p)^x$  with  $x = 3$  (WSA) and 1 (WSB).

The clustering coefficient as a function of the degree

is another important quantity in a network. We find that in the WS networks, the maximum degree is about 10, the clustering coefficients show an exponential decay with  $k$  within this small range of  $k$  (Fig. 4). For WSA the exponential decay becomes much weaker as  $p \rightarrow 1$ , while for WSB, the slope in a log-linear plot shows a much stronger dependence of  $C(k)$  on  $k$  even for large value of  $p$ . The clustering coefficients rapidly decrease as  $p$  increases, as is expected, and at  $p = 1$  becomes very close to zero showing no dependence on  $k$ . The clustering coefficients are obviously larger in WSB than in WSA in magnitude (Fig. 4) and do not show any dependence on the system size for either model.

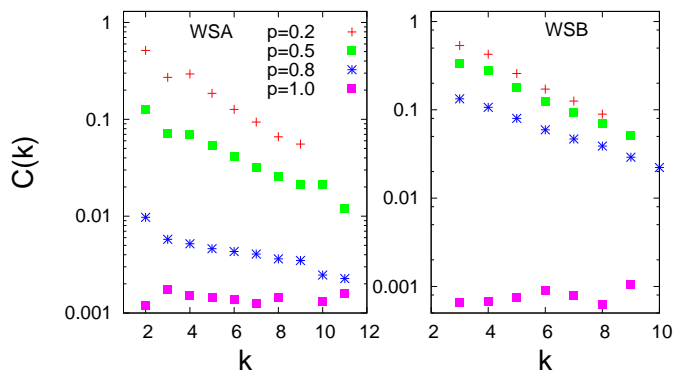


FIG. 4: (Color online) Average clustering co-efficient as a function of the degrees ( $C(k)$  vs  $k$ ) for different rewiring probabilities for the WSA and WSB model for system size  $L = 2000$ .

The degree distribution, which at  $p = 0$  is a delta function, shows a spread as  $p$  is made nonzero. The variance of the degree distribution as a function of  $p$  can be studied. The results show that the nature of variance is identical in WSA and WSB, it shows a monotonic increase with  $p$  as expected [Fig. 5].

One obvious difference between the WSA and WSB regarding the degree distribution is that  $P(k \leq 1) = 0$  for WSA and  $P(k \leq 2) = 0$  for WSB because of the way the rewiring is done [1]. On the other hand the other probabilities, e.g.,  $P(k > 2)$  for WSB are nonzero and almost independent of the system sizes considered.

#### IV. EUCLIDEAN NETWORKS

In the Euclidean network, we have similarly two models which differ in a subtle manner as described in section II.

In Fig. 6, we present the result for the shortest paths against  $L$  for both EA and EB models for different  $\alpha$ . The log-linear plot shows that there could be a deviation from the behaviour  $\langle s \rangle \propto \log L$  even below  $\alpha = 2.0$ . This is in consistency with some earlier results [11, 12, 14] which

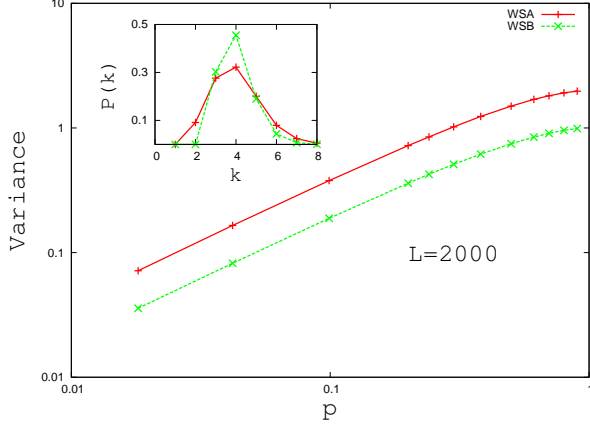


FIG. 5: (Color online) Variance of the degree distribution as a function of rewiring probability  $p$  for the WSA and WSB model for system size  $L = 2000$  are shown. Inset shows the degree distribution  $P(k)$  against  $k$ .

claim that there is another phase transition at  $\alpha = 1$ . We have not attempted a very detailed study to check exactly for which value of  $\alpha$  the small world feature (i.e.,  $\langle s \rangle \propto \log L$ ) vanishes as one needs huge system sizes for that and exact numerical calculations becomes difficult.

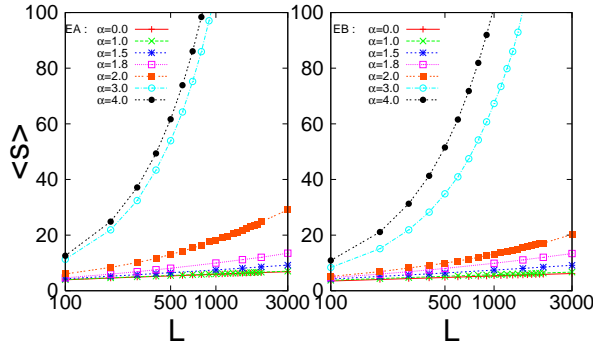


FIG. 6: (Color online) Log-linear plot of variation of average shortest path against system size for different values of  $\alpha$  for EA and EB. The behaviour of  $\langle s \rangle$  apparently deviates from  $\langle s \rangle \propto \log L$  even below  $\alpha = 2.0$ .

When we plot the shortest paths against  $\alpha$ , it is apparent that there is a transition at  $\alpha = 2$  beyond which both EA and EB behave as regular networks. This is shown in Fig. 7. This result is well established in all previous studies. When we compare the shortest paths for EA and EB, we find that finite differences in the results appear for  $2 < \alpha < 10$  and the transition at  $\alpha = 2$  looks much sharper in EA. At large values of  $\alpha$ , for both EA and EB, the shortest paths saturate at a value corresponding to that obtained from (1) as expected.

The average clustering coefficients calculated from these networks are shown as functions of  $\alpha$  in Fig 8. Once again, the values saturate at large values of  $\alpha$  to the ex-

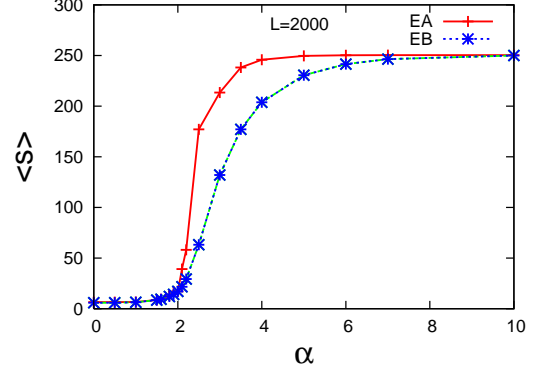


FIG. 7: (Color online) Comparison of average shortest path against  $\alpha$  for EA and EB. In the region  $2 < \alpha < 10$  we have finite differences in  $\langle s \rangle$ .

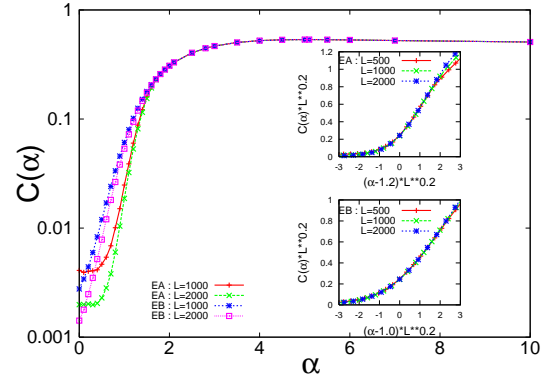


FIG. 8: (Color online) Variation of average clustering coefficients with  $\alpha$  for  $L = 1000$  and  $L = 2000$  are shown for EA and EB models. Insets show the scaling plots for EA (top) and EB (bottom).

pected value  $1/2$ .

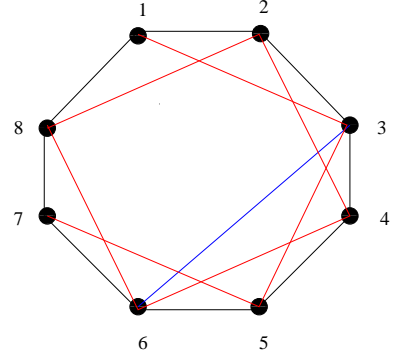


FIG. 9: (Color online) A toy model of EB type in which the average degree is four. The clustering coefficient of this model is  $\simeq 0.53$ , larger than 0.5, the clustering coefficient of a regular network.

There are, however, several interesting facts to be noted. First, we find that clustering coefficient  $C(\alpha)$  is marginally larger than  $1/2$  for some values of  $\alpha > 2$  indicating that a network with some randomness can have

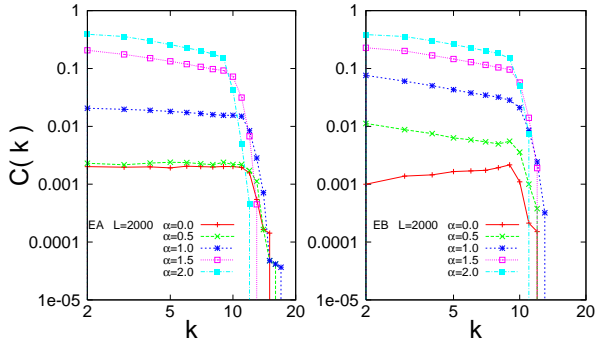


FIG. 10: (Color online) Variation of clustering coefficients against degree for EA and EB for  $L = 1000$ .

larger clustering coefficients compared to the corresponding regular one. We present in Fig. 9 a toy model which has larger clustering coefficient compared to the regular one to show that it can be possible.

In the Euclidean models, the system size dependence of the clustering coefficient shows an interesting result. Normally, one does not expect that clustering coefficient will have size dependence but we do get such a result here in these type of networks. According to [12], it is indeed possible if there is a transition point below which the clustering is zero and above which it is finite. Hence, as in [12], we attempt a data collapse by scaling the variables assuming  $C(\alpha)$  to satisfy the following behaviour

$$C(\alpha) \propto L^{-\gamma} f[(\alpha - \alpha_c)L^\delta], \quad (5)$$

where the scaling function is a constant for very small values of the argument and varies as  $[(\alpha - \alpha_c)L^\delta]^{\gamma/\delta}$  in the opposite limit.

Using the above form, we find excellent data collapse for different sizes with  $\gamma = \delta \simeq 0.2$  for both EA and EB (see insets of Fig. 8). However, the value of  $\alpha_c$  is different;  $\alpha_c \simeq 1.2$  for EA and  $\alpha_c \simeq 1$  for EB. The above analysis gives us a lot of important information: the exponents are independent of EA and EB while the transition points are not. The scaling functions are also different for the two. The values of  $\gamma$  and  $\delta$  being almost same showing that the clustering is a linear function of  $\alpha - \alpha_c$  at  $\alpha \gg \alpha_c$ .

As a function of the degree  $k$ , the clustering coefficients are plotted against  $k$  in Fig. 10. The values of  $k$  are limited and  $C(k)$  shows weak dependence on  $k$  and the variations do not show any drastic change with  $\alpha$  in either EA or EB models.

Next we consider the degree distributions. For both EA and EB, these are shown in Fig. 11. For EA, minimum value of the degree can be zero while in EB, there is a lower cutoff at 2. Consequently, the degree distributions, both of which are peaked around the average value 4, have different variances as function of  $\alpha$  which are plotted in the inset. Here we find that the variance

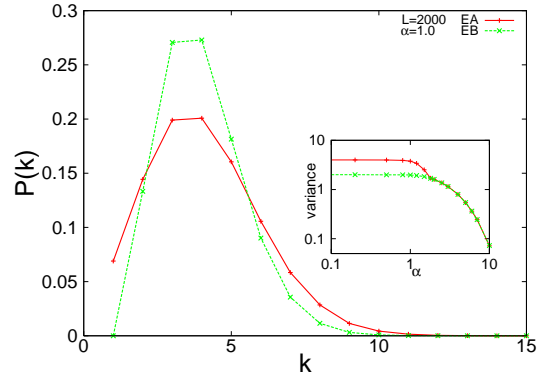


FIG. 11: (Color online) Comparison of degree distribution of EA and EB for system size  $L = 2000$ .

for the two models are significantly different upto  $\alpha \simeq 2$ ; for EB, it is constant upto  $\alpha = 1$  and falls off smoothly, but for EA, it has a different constant value till  $\alpha = 1$ , develops a kink between  $\alpha = 1$  and 2 before converging with the EB values beyond  $\alpha = 2$ .

## V. SUMMARY AND DISCUSSIONS

We considered network models with a fixed average degree  $2K = 4$  where the randomness in linking the nodes has been conceived in different ways.

The WSA and WSB networks, as demonstrated in section III, do not show qualitative differences in the main features. Finite differences in some exponents are there; e.g., the value of  $b$  appearing in the shortest path expression shows a difference. It is expected that for WSA the value of  $b$  should approach unity in the thermodynamic limit. Similarly, for WSB too the value may approach unity for infinite systems. However for the finite sizes considered here, we get different values for WSA and WSB neither of which is equal to unity. Since we evaluate the shortest paths exactly (numerically) it is difficult to study very large sizes. Hence the results may have finite size effects, and if  $b$  really approaches unity for both WSA and WSB in the thermodynamic limit, one can conclude that the finite size effects are more for WSB.

The average clustering coefficient vanishes as  $p \rightarrow 1$  in both WS types of models with an exponent that is markedly different and can be explained why. The variance of the degree distribution on the other hand shows difference in the magnitude only. The clustering coefficients as functions of degree are qualitatively similar in WSA and WSB models although in the latter, the dependence on  $k$  is stronger as  $p \rightarrow 1$ .

For the EA and EB models also, we find the qualitative results to be similar. The presence of the phase transition at  $\alpha = 2$  is evident in both models, the shortest paths in EA showing a sharper transition. Once again, we have estimated the shortest paths exactly which restricts the system sizes considerably such that the position of the second transition (below  $\alpha = 2$ ), if any, is difficult to

establish numerically. However, analysis of the clustering coefficients indicate the presence of another transition at  $\alpha_c$ , the exact value of  $\alpha_c$  being different in EA and EB. The exponents, however, are identical and comparable to the model with a lower average degree studied earlier [12].

One interesting point to note is that apart from the shortest paths, no other quantity shows any difference for EA and EB beyond  $\alpha = 2$ . Although the clustering and degree distribution for EA and EB are identical for  $\alpha > 2.0$ , there is an interesting behaviour of the clustering coefficient which increases beyond the value 0.5 corresponding to the regular network.

The Euclidean model B has been considered to be appropriate to model linear polymers or self avoiding walks [10, 11]. The value of  $\alpha$  which will be comparable to a linear polymer is less than 2. However, the absence of magnetic phase transition on a polymer chain indicates that it behaves as a regular network while for the Euclidean model B, the behaviour at  $\alpha < 2$  is definitely different from that of a linear chain. This indicates that the EB does not really mimic the behaviour of a linear polymer possibly because in the former, the long ranged links are uncorrelated which is not true for a polymer chain where the bridges are correlated.

As mentioned in section II, the WS and Euclidean models are equivalent at extreme limiting values of the parameter  $\alpha$  and  $p$ . So it also makes sense to compare the two when the parameters assume intermediate values. The main difference is of course that there is at least one non-trivial phase transition in the Euclidean models. Increasing  $\alpha$  beyond 2, we have an extensive region where regular network-like behaviour can be observed. No such study is possible for the WS type networks, which behaves like a small world network for any nonzero value of  $p \neq 1$ . In case one concludes that the Euclidean networks have a random nature below  $\alpha_c$ , then there is a finite re-

gion  $0 < \alpha < \alpha_c$  with random network behaviour. For the WS type networks, random network behaviour exists for  $p = 1$  only.

In both models, the degree distributions have exponential tails. The variance in the Euclidean network however shows a constant value over a considerable region, which is not observed in the WS models. This may be due to the random network like behaviour of the Euclidean models in the entire region  $\alpha < \alpha_c$ .

The clustering coefficient as function of the parameter  $p$  or  $\alpha$  shows that it has a scaling behaviour as the point  $p = 1$  or  $\alpha = \alpha_c$  is approached. While for WS type networks, the exponents are appreciably different for the different schemes of rewiring, for the Euclidean networks, the two exponents  $\gamma$  and  $\delta$  are identical for EA and EB while the value of  $\alpha_c$  shows a difference.

For WSA each node has at least  $K/2 = 2$  edges after the rewiring process [1]. So there are no isolated nodes and the network is usually connected. But for EA there is no such restriction and one gets broken networks for EA with much higher probability. In our simulations broken WSA networks were not obtained at all (unless  $L$  is very small) while for the EA, such networks, even for large values of  $L$  were generated and were not considered for the various calculations.

In summary, we have made a detailed study of different properties of some models of networks in which the randomness has been incorporated in different ways. We conclude that small changes in the randomness in network models do not alter the gross qualitative features.

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